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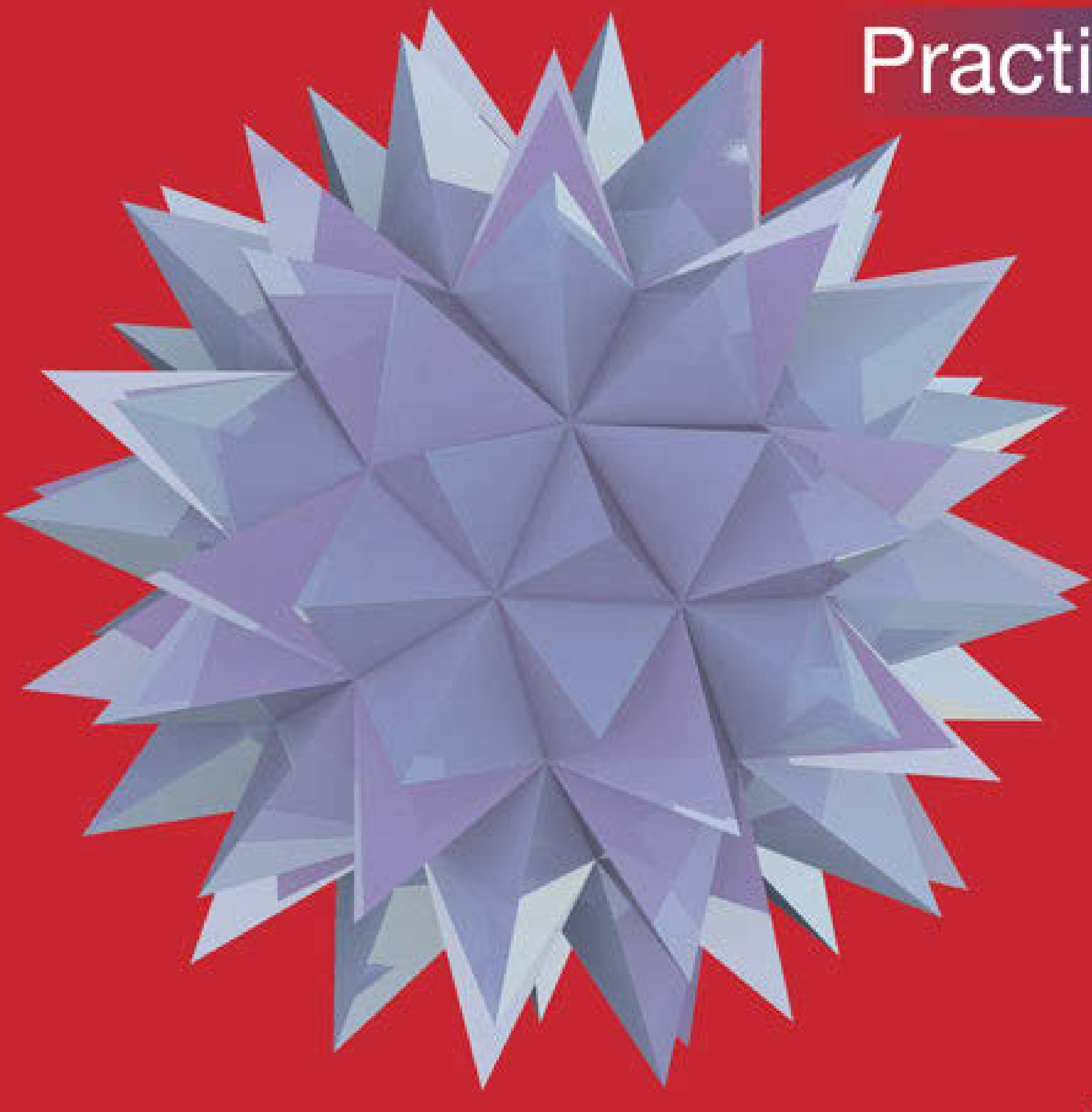
Muriel James

Cambridge IGCSE® and O Level Additional Mathematics

Practice Book



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Contents

Introduction

This practice book offers full coverage of the *Cambridge IGCSE* and *O Level Additional Mathematics* syllabuses (0606 and 4037). It has been written by a highly experienced author, who is very familiar with the syllabus and the examinations. The course is aimed at students who are currently studying or have previously studied *Cambridge IGCSE Mathematics* (0580) or *Cambridge O Level Mathematics* (4024).

The practice book has been written to closely follow the chapters and topics of the coursebook offering additional exercises to help students to consolidate concepts learnt.

At the start of each chapter, there is a list of objectives that are covered in the chapter. These objectives have been taken directly from the syllabus.

Worked examples are used throughout to demonstrate the methods for selected topics using typical workings and thought processes. These present the methods to the students in a practical and easy-to-follow way that minimises the need for lengthy explanations.

The exercises are carefully graded. They offer plenty of practice via 'drill' questions at the start of each exercise, which allow the student to practice methods that have just been introduced. The exercises then progress to questions that typically reflect the kinds of questions that the student may encounter in the examinations.

Towards the end of each chapter, there is a summary of the key concepts to help students consolidate what they have just learnt. This is followed by an 'exam-style' questions section to assess their learning after each chapter.

The answers to all questions are supplied at the back of the book, allowing self- and/or class-assessment. A student can assess their progress as they go along, choosing to do more or less practice as required. The answers given in this book are concise and it is important for students to appreciate that in an examination they should show as many steps in their working as possible.

A Coursebook is available in the *Additional Mathematics* series, which offers comprehensive coverage of the syllabus. This book includes class discussion activities, worked examples for every method, exercises and a 'Past paper' questions section, which contains real questions taken from past examination papers. A Teacher's resource CD-ROM, to offer support and advice, is also available.

How to use this book

Chapter

Each chapter begins with a set of learning objectives to explain what you will learn in the chapter.

Chapter 2:
Functions

This section will show you how to:

- understand and use the terms: function, domain, range (image set), one-one function, inverse function and composition of functions
- use the notation $f(x) = 2x^3 + 5$, $f : x \mapsto 5x - 3$, $f^{-1}(x)$ and $f^2(x)$
- understand the relationship between $y = f(x)$ and $y = |f(x)|$
- explain in words why a given function is a function or why it does not have an inverse
- find the inverse of a one-one function and form composite functions
- use sketch graphs to show the relationship between a function and its inverse.

Reminder

At the start of each chapter, a Reminder box reminds you of key concepts from the corresponding chapter in the Coursebook. If you are unsure of any of these concepts, look back at the chapter in the Coursebook.

REMINDER

- When one function is followed by another function, the resulting function is called a **composite function.**
- $fg(x)$ means the function g acts on x first, then f acts on the result.
- $f^2(x)$ means $ff(x)$, so you apply the function f twice.

Remember

Remember boxes contain equations or formulae that you need to know.

REMEMBER

$(x - \sqrt{y})(x + \sqrt{y}) = x^2 - y$

Worked Example

Detailed step-by-step approaches to help students solve problems.

WORKED EXAMPLE 1

Solve the simultaneous equations.

$x + 2y = 4$

$x^2 + 4y^2 = 10$

Answers

$x + 2y = 4$ --------------(1)

$x^2 + 4y^2 = 10$ ---------(2)

From (1), $x = 4 - 2y$.

Substitute for x in (2):

$(4 - 2y)^2 + 4y^2 = 10$ expand brackets

$16 - 16y + 4y^2 + 4y^2 = 10$ rearrange

$8y^2 - 16y + 6 = 0$ simplify

$4y^2 - 8y + 3 = 0$ factorise

$(2y - 1)(2y - 3) = 0$

$y = \frac{1}{2}$ or $y = 1\frac{1}{2}$

Substituting $y = \frac{1}{2}$ into (1) gives $x = 3$.

Substituting $y = 1\frac{1}{2}$ into (1) gives $x = 1$.

The solutions are: $x = 1,\ y = 1\frac{1}{2}$ and $x = 3,\ y = \frac{1}{2}$.

Tip

Tip boxes contain helpful hints for working through questions.

TIP

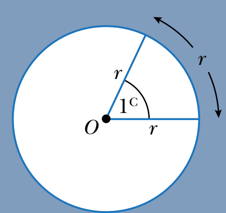
Squaring is not a reversible step. Notice that $x = 16$ does **not** satisfy the original equation.

Summary

At the end of each chapter to review what you have learned.

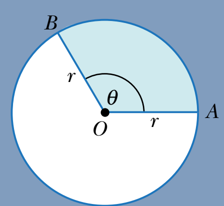
Summary

One radian (1^c) is the size of the angle subtended at the centre of a circle, radius r, by an arc of length r.



When θ is measured in radians:

- the length of arc $AB = r\theta$
- the area of sector $AOB = \frac{1}{2}r^2\theta$



Chapter 1: Sets

This section will show you how to:

- use set language and notation and Venn diagrams to describe sets and represent relationships between sets.

1.1 The language of sets

REMINDER

- You should already be familiar with the following set notation:

  $A = \{x : x \text{ is a natural number}\}$
  $B = \{(x, y): y = mx + c\}$
  $C = \{x: a \leqslant x \leqslant b\}$
  $D = \{a, b, c, ...\}$

- You should be familiar with the following set symbols:

Union of A and B	$A \cup B$	The empty set	$\varnothing$
Intersection of A and B	$A \cap B$	Universal set	$\mathscr{E}$
Number of elements in set A	$\text{n}(A)$	A is a subset of B	$A \subseteq B$
'... is an element of...'	$\in$	A is a proper subset of B	$A \subset B$
'... is no an element of...'	$\notin$	A is not a subset of B	$A \nsubseteq B$
Complement of set A	A'	A is not a proper subset of B	$A \not\subset B$

- The set of natural numbers $\{1, 2, 3, ...\}$ $\mathbb{N}$ The set of integers $\{0, \pm1, \pm2, \pm3, ...\}$ $\mathbb{Z}$
  The set of real numbers $\mathbb{R}$ The set of rational numbers $\mathbb{Q}$

WORKED EXAMPLE 1

$\mathscr{E} = \{x : 2 \leqslant x \leqslant 10, x \text{ is a integer}\}$
$A = \{x : 5 \leqslant x \leqslant 8\}$
$B = \{x : x > 5\}$
$C = \{x : x \text{ is a factor of } 6\}$

List the elements of
a $A \cap B$ **b** $(A \cap B)'$ **c** $(A \cap B)' \cap C$.

Answers

List the elements of each set: $A = \{5, 6, 7, 8\}$, $B = \{6, 7, 8, 9, 10\}$ and $C = \{2, 3, 6\}$

a $A \cap B = \{6, 7, 8\}$ Elements in both A and B.
b $(A \cap B)' = \{2, 3, 4, 5, 9, 10\}$ Elements not in $A \cap B$.
c $(A \cap B)' \cap C = \{2,3\}$

Exercise 1.1

1 $\mathscr{E} = \{x: 3 < x < 14,\ x \text{ is a integer}\}$

$A = \{x: 5 < x \leqslant 12\}$

List the elements of A'.

2 $\mathscr{E} = \{x: 0 < x < 12,\ x \text{ is a integer}\}$

$P = \{x : x^2 - 5x + 4 = 0\}$

$Q = \{x : 2x - 4 < 2\}$

Find the values of x such that

a $x \in P$ **b** $x \in Q$ **c** $x \in P \cap Q$ **d** $x \in (P \cup Q)'$.

3 $\mathscr{E}$ = {positive integers less than 10}

$P = \{3, 4, 5, 6, 7, 8\}$

$P \cap Q = \varnothing$

Write down all the possible members of Q.

4 $\mathscr{E}$ = {members of an outdoor pursuits club}

S = {members who go swimming}

R = {members who go running}

W = {members who go walking}

Write the following statements using set notation.

a There are 65 members of the club.

b There are 31 members who go running.

c There are 15 members who go running and swimming.

d Every member that goes running also goes walking.

5 $\mathscr{E}$ = {members of a sports club} T = {members who like tennis}

R = {members who like rowing} S = {members who like squash}

Describe the following in words

a $T \cup R$ **b** $T \cap S$ **c** R' **d** $R \cap S = \varnothing$.

6 $\mathscr{E}$ = {students in a school} C = {students studying chemistry}

M = {students studying mathematics} P = {students studying physics}

a Express the following statements using set notation

i all physics students also study mathematics

ii no student studies both chemistry and physics.

b Describe the following in words

i $C \cap M \cap P'$ **ii** $C' \cap (M \cup P)$.

TIP [6A PART (ii)]
There are two ways of expressing this!

7 $\mathscr{E} = \{x: 25 \leqslant x \leqslant 75, \text{ where } x \text{ is an integer}\}$ C = {cube numbers}

P = {prime numbers} S = {square numbers}

Express the following statements using set notation

a 29 is a prime number

b 32 is not a cube number

c there are 2 cube numbers between 25 and 75 inclusive

d there are 12 integers between 25 and 75 inclusive, that are prime

e there are no cube numbers that are prime.

8 $\mathscr{E}$ = {students in a school} B = {students in the basketball team}

N = {students in the netball team} S = {students in the swimming team}

G = {students who are girls}

Express the following statements using set notation

a all students in the netball team are girls

b all students in the swimming team are boys

c there are no students who are in both the basketball team and the netball team

d there are 5 people who are in both the basketball team and the swimming team.

9 Illustrate each of the following sets on a graph.

a $\{(x, y) : y = 3x + 1\}$ **b** $\{(x, y) : x + y = 8\}$ **c** $\{(x, y) : y = 4\}$

10 $A = \{(x, y) : y = 3x + 5\}$ $B = \{(x, y) : y = 2\}$

$C = \{(x, y) : x + y = 9\}$ $D = \{(x, y) : y = 2x - 2\}$

a List the elements of

i $A \cap B$ **ii** $A \cap C$.

b Find

i $n(B \cap D)$ **ii** $n(A \cap D)$.

11 In each of the following sets, $x \in \mathbb{R}$.

$A = \{x : 8 - 3x = 11\}$ $B = \{x : x^2 - x - 6 = 0\}$

$C = \{x : x^2 + 4x + 4 = 0\}$ $D = \{x : x(x + 2)(x - 3) = 0\}$

$E = \{x : x^2 - x - 12 = 0\}$

a Find

i $n(A)$ **ii** $n(B)$ **iii** $n(C)$ **iv** $n(D)$.

b List the elements of the sets

i $B \cup D$ **ii** $B \cap D$.

c Use set notation to complete the statement: $C \cap E = \ldots$

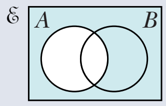
1.2 Shading sets on Venn diagrams

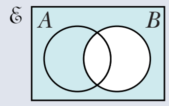
WORKED EXAMPLE 2

On a Venn diagram shade the regions:

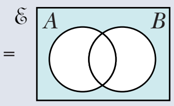
a $A' \cap B'$ **b** $A' \cup B'$

Answers

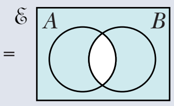
$A' =$ 

$B' =$ 

a $A' \cap B'$ is the region that is in both A' and B.

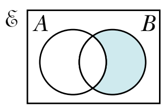
$A' \cap B' =$ 

b $A' \cup B'$ is the region that is in A' or B' or both so you need all the shaded regions.

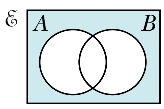
$A' \cup B' =$ 

Exercise 1.2

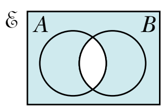
1 Match each picture with two of the descriptions below (one from each)

A 

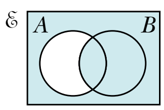
i $A \cap B$ **a** $A' \cap B'$

B 

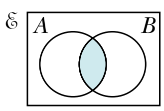
ii $A' \cap B$ **b** $(A \cap B')'$

C 

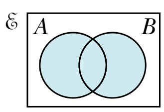
iii $A \cup B$ **c** $(A' \cup B')'$

D 

iv $A \cup B'$ **d** $(A \cup B')'$

E 

v $(A \cup B)'$ **e** $(A' \cap B')'$

F 

vi $A' \cup B$ **f** $(A \cap B)'$

2 $\mathscr{E}$ = {polygons} S = {squares} Q = {quadrilaterals} R = {rectangles}

a Represent these sets on a Venn diagram.

b Write true or false for the following statements :

i $S \subset Q$ .... **ii** $R \subset S$ ....

iii $S \cap R = S$ .... **iv** $Q' = \varnothing$ ....

c On your diagram, shade the region represented by $R' \cap Q$.

3 For each of the statements below, write 'true' or 'false'.

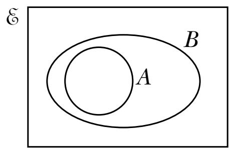
a $\{7, 11\} \subset \{2, 5, 7, 11, 13\}$ **b** $7 \in \{7, 11\}$

c $\{7, 11\} \cap \{2, 3, 7\} = \varnothing$ **d** $\{1, 2, 3\} \cap \{4, 6, 7\} = \{1, 2, 3, 4, 6, 7\}$

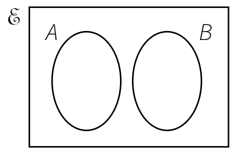
4 Choose a statement which describes the relationship between sets A and B for each diagram.

Statements: **1** $A \cap B = B$ **2** $B \subset A$ **3** $A \subset B$ **4** $A \cap B = \varnothing$ **5** $A \cup B = \mathscr{E}$

a



b



1.3 Describing sets on a Venn diagram

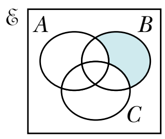
Exercise 1.3

1 $\mathscr{E}$ = {students in a class} A = {students studying art}

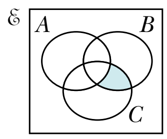
B = {students studying biology} C = {students studying chemistry}

Match each worded statement with a diagram, e.g. **a ii**.

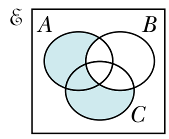
a



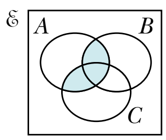
b



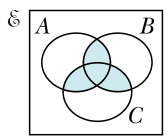
c



d



e



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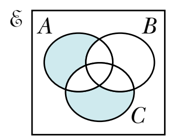


i Students who study art and/or chemistry but not biology.

ii Students who study at least two of these subjects.

iii Students who study biology only.

iv Students who study only one of the subjects but not biology.

v Students who study biology and chemistry but not art.

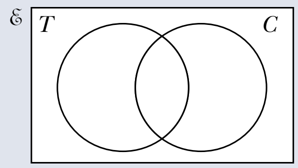
vi Students who study art with biology and/or chemistry.

1.4 Number of elements in regions on a Venn diagram

WORKED EXAMPLE 3

In a class of 23 students, 15 like coffee, 13 like tea and 4 students don't like either drink.

a Copy the Venn diagram and enter the information.



Use it to find how many students like:

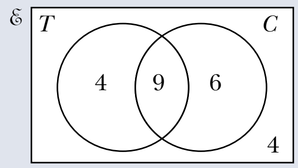
b tea only

c coffee only

d both drinks

Answers

a Draw a Venn diagram and put 4 outside T and C.



There are a total of 23 students so $n(T \cup C) = 23 - 4 = 19$

$n(T \cap C) = n(T) + n(C) - n(T \cup C)$

$n(T \cap C) = 13 + 15 - 19$

$n(T \cap C) = 9$ Put 9 in the intersection.

There are a total of 13 items in T, so 4 must go in the remaining part of T.

There are a total of 15 items in C, so 6 must go in the remaining part of C.

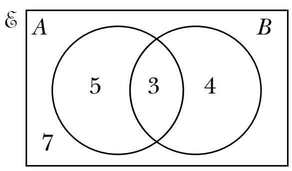
b $n(T \cap C') = 4$

c $n(C \cap T') = 6$

d $n(T \cap C) = 9$

Exercise 1.4

1 The numbers in the diagram represent the number of people.



Find:

a $n(A)$ **b** $n(B')$ **c** $n(A \cap B \cap B')$ **d** $n(A' \cup B')$

2 In a class of 40 students, 18 watched a rugby match, 23 watched a football match and 7 watched both matches. How many students did not watch either match?

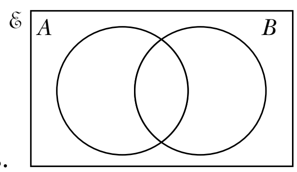
3 In one class in a school, 13 students are studying drama, 12 students are studying art and 8 students are studying both. Five students are studying neither. How many students are there in the class?

4 There are 100 spectators at an ice hockey match. 79 are wearing a hat, 62 are wearing a scarf and 27 are wearing a hat but not a scarf. How many are wearing neither a hat nor a scarf.

5 $n(\mathscr{E}) = 35$ $n(A) = 21$

$n(B) = 17$ $n(A' \cap B) = 8$

In a copy of the Venn diagram insert the numbers of elements in the set represented by each of the four regions.



6 There are 31 members of a sports club, 15 like playing basketball, 13 like playing football and 8 don't like playing either sport. How many like playing:

a basketball only? **b** football only? **c** both sports?

7 52 students are going on a skiing trip. 28 have skied before, 30 have snowboarded before and 12 have done neither before. How many have done both sports before?

8 In a class with 40 students, 18 study art and 7 study art and music. Six do not study either. How many study music?

The following exercise involves problem solving.

Start each question by displaying the given information on a Venn diagram.

Then use your Venn diagram to solve the problem.

Exercise 1.5

1 In a sports club, 68 members play football, 72 members play hockey and 77 play basketball. 44 play football and hockey, 55 play hockey and basketball, 50 play football and basketball and 32 members play all three sports. How many members are there in this club?

TIP
Show the information on a Venn diagram.

2 In a class of 30 students, 16 students study French, 16 want to take Spanish, 11 want to take German. Five students take both French and German, and of these, 3 want to take Spanish as well. Five want to study German only and 8 want Spanish only. How many students want French only?

3 In a class of 40 students, 19 study art, 18 study drama and 20 study music. Eleven students study art and music, 5 study art and drama and 8 study music and drama. 5 students do not study any of the three subjects.

How many students study:

a all three activities?

b exactly two activities?

c music only?

4 In a class of 25 students, 19 have watched film A and 14 have watched film B. x students have watched both films and y students have watched neither film. What are the largest and smallest possible values of x and y?

Summary

The language of sets

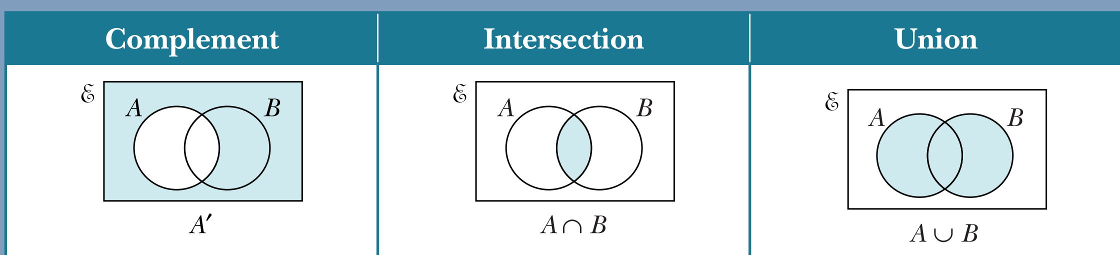
Union of A and B	$A \cup B$	The empty set	$\varnothing$
Intersection of A and B	$A \cap B$	Universal set	$\mathscr{E}$
Number of elements in set A	$\mathrm{n}(A)$	A is a subset of B	$A \subseteq B$
'… is an element of…'	$\in$	A is a proper subset of B	$A \subset B$
'… is not an element of…'	$\notin$	A is not a subset of B	$A \nsubseteq B$
Complement of set A	A'	A is not a proper subset of B	$A \not\subset B$

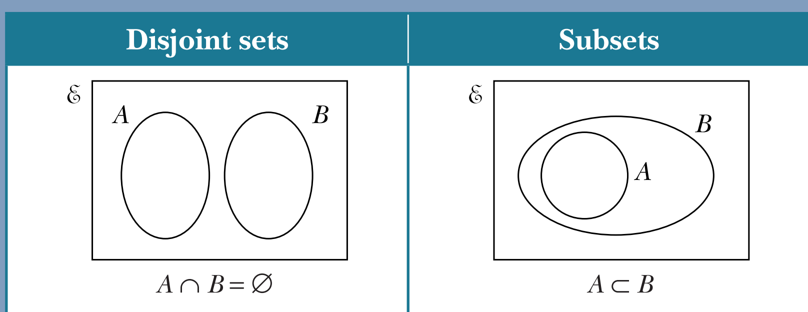
Some special number sets

The set of natural numbers {1, 2, 3, …}	$\mathbb{N}$
The set of integers {0, ±1, ±2, ±3, …}	$\mathbb{Z}$
The set of real numbers	$\mathbb{R}$
The set of rational numbers	$\mathbb{Q}$

Venn diagrams

You should also know how to represent the complement, union and intersections of sets on a Venn diagram:





Useful rules to remember

$$(A \cap B)' = A' \cup B'$$

$$(A \cup B)' = A' \cap B'$$

$$\mathrm{n}(A \cup B) = \mathrm{n}(A) + \mathrm{n}(B) - \mathrm{n}(A \cap B)$$

Exercise 1.6

1 $\mathscr{E} = \{x : 3 \leqslant x \leqslant 11, x \in \mathbb{R}\}$

$A = \{x : 2 < x \leqslant 8\}$

$B = \{x : 6 < x < 10\}$

Find the sets

a A' **b** B' **c** $A \cap B$ **d** $A \cup B$.

2 $\mathscr{E}$ = {students in a school}

A = {students who are over 190 cm tall}

B = {students who wear coats}

C = {students who bike to school}

a Express each of the following in words

i $A \cap C \neq \varnothing$

ii $A \subset B'$

b Express in set notation the statement 'all students who are over 190 cm tall and who bike to school do not wear coats'.

3 $A = \{x : 2x^2 + x - 6 \leqslant 0\}$ $B = \{x : 2 - 2x \leqslant 4\}$

a Find the set of values which define the set A.

b Find the set of values of x which define the set $A \cap B$.

4 $\mathscr{E}$ = {students in a class}

P = {students studying physics}

B = {students studying biology}

C = {students studying chemistry}

All students who study biology also study chemistry.

i Represent these three sets on a Venn diagram, then shade the region which represents those students who study biology and chemistry but not physics.

ii On a second copy of your diagram, shade those students who study chemistry only.

5 There are 40 students in a class.

F = {students who study French}

G = {students who study German}

$n(F) = 23$ $n(G) = 15$ $n(F \cap G) = x$ $n(F' \cap G') = 4$

Write down an equation in x and hence find the number of students in the group who study German only.

TIP Q4

Always assume that there is an intersection of all 3 sets unless, from reading the question, you interpret otherwise.

When filling in your Venn diagram, start with the overlap of all 3 sets, then the overlap of two sets, etc.

6 $n(A) = 14$ $n(B) = 20$

$n(C) = 12$ $n(A \cap B) = 10$

$n(A \cap C) = 6$ $n(B \cap C) = 7$

$n(A \cap B \cap C) = 3$ $n(A \cup B \cup C)' = 6$

Draw a Venn diagram to find $n(\mathscr{E})$.

7 In a year group of 70 pupils, F is the set of pupils who play football, and H is the set of pupils who play hockey. There are 40 pupils who play football and 30 who play hockey.

The number who play both sports is x and the number who play neither is $12 - 2x$.

Find:

a the value of x

b $n(F \cap H')$.

Chapter 2: Functions

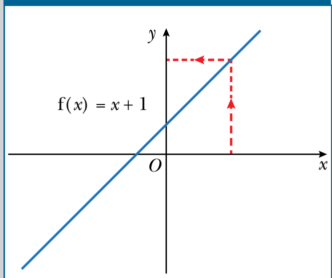
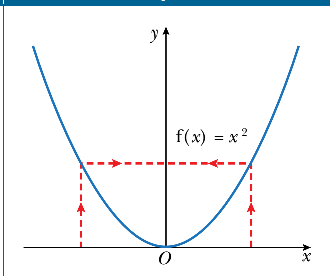
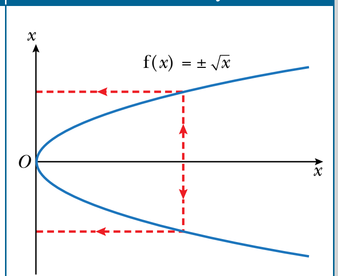
This section will show you how to:

- understand and use the terms: function, domain, range (image set), one-one function, inverse function and composition of functions
- use the notation $f(x) = 2x^3 + 5$, $f : x \mapsto 5x - 3$, $f^{-1}(x)$ and $f^2(x)$
- understand the relationship between $y = f(x)$ and $y = |f(x)|$
- explain in words why a given function is a function or why it does not have an inverse
- find the inverse of a one-one function and form composite functions
- use sketch graphs to show the relationship between a function and its inverse.

2.1 Mappings

REMINDER

The table below shows one-one, many-one and one-many mappings.

one-one	many-one	one-many
		
For one input value there is just one output value.	For two input values there is one output value.	For two input value there are two output values.

Exercise 2.1

Determine whether each of these mappings is one-one, many-one or one-many.

1 $x \mapsto 2x + 3 \quad x \in \mathbb{R}$

2 $x \mapsto x^2 + 4 \quad x \in \mathbb{R}$

3 $x \mapsto 2x^3 \quad x \in \mathbb{R}$

4 $x \mapsto 3^x \quad x \in \mathbb{R}$

5 $x \mapsto \frac{-1}{x} \quad x \in \mathbb{R},\ x > 0$

6 $x \mapsto x^2 + 1 \quad x \in \mathbb{R},\ x \geqslant 0$

7 $x \mapsto \frac{2}{x} \quad x \in \mathbb{R},\ x > 0$

8 $x \mapsto \pm\sqrt{x} \quad x \in \mathbb{R},\ x \geqslant 0$

2.2 Definition of a function

REMINDER

A function is a rule that maps each x value to just one y value for a defined set of input values.

This means that mappings that are either $\left\{\begin{array}{l}\text{one-one}\\ \text{many-one}\end{array}\right.$ are called functions.

The mapping $x \mapsto x + 1$ where $x \in \mathbb{R}$, is a one-one function.

The function can be defined as $\text{f}: x \mapsto x + 1, \quad x \in \mathbb{R}$ or $\text{f}(x) = x + 1, \quad x \in \mathbb{R}$.

The set of input values for a function is called the **domain** of the function.

The set of output values for a function is called the **range** (or image set) of the function.

WORKED EXAMPLE 1

The function f is defined by $\text{f}(x) = (x-1)^2 + 4$ for $0 \leqslant x \leqslant 5$.

Find the range of f.

Answers

$\text{f}(x) = (x-1)^2 + 4$ is a positive quadratic function so the graph will be of the form

$(x-1)^2 + 4$

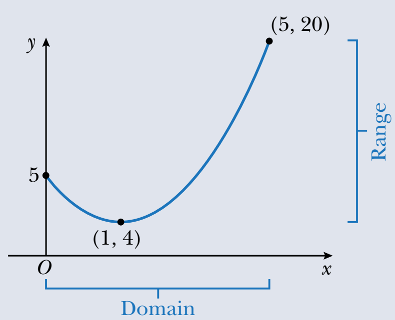
This part of the expression is a square so it will always be $\geqslant 0$.
The smallest value it can be is 0. This occurs when $x = 1$.

The minimum value of the expression is $0 + 4 = 4$ and this minimum occurs when $x = 1$.

So the function $\text{f}(x) = (x-1)^2 + 4$ will have a minimum point at the point $(1, 4)$.

When $x = 0$, $y = (0-1)^2 + 4 = 5$.

When $x = 5$, $y = (5-1)^2 + 4 = 20$.



The range is $1 \leqslant \text{f}(x) \leqslant 20$.

Exercise 2.2

1 Which of the mappings in **Exercise 2.1** are functions?

2 Find the range for each of these functions.

a $\text{f}(x) = x - 9, \quad -2 \leqslant x \leqslant 8$

b $\text{f}(x) = 2x - 2, \quad 0 \leqslant x \leqslant 6$

c $\text{f}(x) = 7 - 2x, \quad -3 \leqslant x \leqslant 5$

d $\text{f}(x) = 2x^2, \quad -4 \leqslant x \leqslant 3$

e $\text{f}(x) = 3^x, \quad -4 \leqslant x \leqslant 3$

f $\text{f}(x) = \dfrac{-1}{x}, \quad 1 \leqslant x \leqslant 6$

3 The function g is defined as $\text{g}(x) = x^2 - 5$ for $x \geqslant 0$.

Find the range of g.

4 The function f is defined by $\text{f}(x) = 4 - x^2$ for $x \in \mathbb{R}$.

Find the range of f.

5 The function f is defined by $f(x) = 3 - (x-1)^2$ for $x \geqslant 1$.
Find the range of f.

6 The function f is defined by $f(x) = (4x+1)^2 - 2$ for $x \geqslant -\frac{1}{4}$.
Find the range of f.

7 The function f is defined by $f : x \mapsto 8 - (x-3)^2$ for $2 \leqslant x \leqslant 7$.
Find the range of f.

8 The function f is defined by $f(x) = 3 - \sqrt{x-1}$ for $x \geqslant 1$.
Find the range of f.

9 Find the largest possible domain for the following functions:

a $f(x) = \frac{1}{x+3}$ **b** $f(x) = \frac{3}{x-2}$ **c** $\frac{4}{(x-3)(x+2)}$

d $f(x) = \frac{1}{x^2-4}$ **e** $f : x \mapsto \sqrt{x^3-4}$ **f** $f : x \mapsto \sqrt{x+5}$

g $g : x \mapsto \frac{1}{\sqrt{x-2}}$ **h** $f : x \mapsto \frac{x}{\sqrt{3-3x}}$ **i** $f : x \mapsto 1 - x^2$

2.3 Composite functions

REMINDER

- When one function is followed by another function, the resulting function is called a **composite function.**
- $fg(x)$ means the function g acts on x first, then f acts on the result.
- $f^2(x)$ means $ff(x)$, so you apply the function f twice.

WORKED EXAMPLE 2

$f : x \mapsto 4x + 3$ for $x \in \mathbb{R}$
$g : x \mapsto 2x^2 - 5$ for $x \in \mathbb{R}$
Find fg(3).

Answer

fg(3) g acts on 3 first and $g(3) = 2 \times 3^2 - 5 = 13$.
= f(13)
$= 4 \times 13 + 3$
= 55

WORKED EXAMPLE 3

$g(x) = 2x^2 - 2$ for $x \in \mathbb{R}$
$h(x) = 4 - 3x$ for $x \in \mathbb{R}$
Solve the equation $hg(x) = -14$.

Answers

$\text{hg}(x)$	g acts on x first and $\text{g}(x) = 2x^2 - 2$.
$= \text{h}(2x^2 - 2)$	h is the function 'triple and take from 4'.
$= 4 - 3(2x^2 - 2)$	Expand the brackets.
$= 4 - 6x^2 + 6$	
$= 10 - 6x^2$	
$\text{hg}(x) = -14$	
$-14 = 10 - 6x^2$	Set up and solve the equation.
$24 = 6x^2$	
$4 = x^2$	
$x = \pm 2$	

Exercise 2.3

1 $\text{f}(x) = 2 - x^2$ for $x \in \mathbb{R}$

$\text{g}(x) = \frac{x}{2} + 3$ for $x \in \mathbb{R}$

Find the value of $\text{gf}(4)$.

2 $\text{f}(x) = (x-2)^2 - 2$ for $x \in \mathbb{R}$

Find $\text{f}^2(3)$.

3 The function f is defined by $\text{f}(x) = 1 + \sqrt{x-3}$ for $x \geqslant 3$.

The function g is defined by $\text{g}(x) = \frac{-3}{x} - 1$ for $x > 0$.

Find gf(7).

4 The function f is defined by $\text{f}(x) = (x-2)^2 + 3$ for $x > -2$.

The function g is defined by $\text{g}(x) = \frac{3x+4}{x+2}$ for $x > 2$.

Find fg(6).

5 $\text{f} : x \mapsto 3x - 1$ for $x > 0$

$\text{g} : x \mapsto \sqrt{x}$ for $x > 0$

Express each of the following in terms of f and g.

a $x \mapsto 3\sqrt{x} - 1$ **b** $x \mapsto \sqrt{3x-1}$

6 The function f is defined by $\text{f} : x \mapsto 2x - 1$ for $x \in \mathbb{R}$.

The function g is defined by $\text{g} : x \mapsto \frac{8}{4-x}$ for $x \neq 4$.

Solve the equation $\text{gf}(x) = 5$.

7 $\text{f}(x) = 2x^2 + 3$ for $x > 0$

$\text{g}(x) = \frac{5}{x}$ for $x > 0$

Solve the equation $\text{fg}(x) = 4$.

8 The function f is defined, for $x \in \mathbb{R}$, by $\text{f} : x \mapsto \frac{2x-1}{x-3}, x \neq 3$.

The function g is defined, for $x \in \mathbb{R}$, by $\text{g} : x \mapsto \frac{x+1}{2} x \neq 1$.

Solve the equation $\text{fg}(x) = 4$.

TIP

Before writing your final answers, compare your solutions with the domains of the original functions.

9 The function g is defined by $g(x) = 1 - 2x^2$ for $x \geqslant 0$.

The function h is defined by $h(x) = 3x - 1$ for $x \geqslant 0$.

Solve the equation $gh(x) = -3$ giving your answer(s) as exact value(s).

10 The function f is defined by $f : x \mapsto x^2$ for $x \in \mathbb{R}$.

The function g is defined by $g : x \mapsto x + 2$ for $x \in \mathbb{R}$.

Express each of the following as a composite function, using only f and g:

a $x \mapsto (x+2)^2$ **b** $x \mapsto x^2 + 2$ **c** $x \mapsto x + 4$ **d** $x \mapsto x^4$

11 The functions f and g are defined for $x > 0$ by $f : x \mapsto x + 3$ and $g : x \mapsto \sqrt{x}$

Express in terms of f and g:

a $x \mapsto \sqrt{x+3}$ **b** $x \mapsto x + 6$ **c** $x \mapsto \sqrt{x} + 3$

12 Given the functions $f(x) = \sqrt{x}$ and $g(x) = \dfrac{x-5}{2x+1}$,

a Find the domain and range of g.

b Solve the equation $g(x) = 0$.

c Find the domain and range of fg.

2.4 Modulus functions

REMINDER

- The **modulus** (or **absolute value**) of a number is the magnitude of the number without a sign attached.
- **The modulus of** x, written as $|x|$, is defined as

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

- **The statement** $|x|, = k$, where $\geqslant 0$, means that $x = k$ or $x = -k$.

WORKED EXAMPLE 4

a $|4x + 3| = x + 18$ **b** $|2x^2 - 9| = 7$

Answers

a $|4x + 3| = x + 18$

$4x + 3 = x + 18$ or $4x + 3 = -x - 18$

$3x = 15$ $\quad 5x = -21$

$x = 5$ $\quad x = -\dfrac{21}{5}$

Solution is : $x = 5$ or $-\dfrac{21}{5}$

b $|2x^2 - 7| = 9$

$2x^2 - 7 = 9$ or $2x^2 - 7 = -9$

$2x^2 = 16$ $\quad 2x^2 = -2$

$x^2 = 8$ $\quad x^2 = -1$

$x = \pm 2\sqrt{2}$

Solution is : $x = \pm 2\sqrt{2}$

Exercise 2.4

> **TIP**
> Remember to check your answers to make sure that they satisfy the original equation.

1 Solve.

a $|2x-1| = 11$ **b** $|2x+4| = 8$ **c** $|6-3x| = 4$

d $\left|\frac{x-2}{5}\right| = 6$ **e** $\left|\frac{3x+4}{3}\right| = 4$ **f** $\left|\frac{9-2x}{3}\right| = 4$

g $\left|\frac{x}{3}-6\right| = 1$ **h** $\left|\frac{2x+5}{3}+\frac{2x}{5}\right| = 3$ **i** $|2x-6| = x$

2 Solve.

a $\left|\frac{2x-5}{x+4}\right| = 3$ **b** $\left|\frac{4x+2}{x+3}\right| = 3$ **c** $\left|1+\frac{2x+5}{x+3}\right| = 4$

d $|2x-3| = 3x$ **e** $2x+|3x-4| = 5$ **f** $7-|1-2x| = 3x$

3 Solve giving your answers as exact values if appropriate.

a $|x^2-4| = 5$ **b** $|x^2+5| = 11$ **c** $|9-x^2| = 3-x$

d $|x^2-3x| = 2x$ **e** $|x^2-16| = 2x+1$ **f** $|2x^2-1| = x+2$

g $|3-2x^2| = x$ **h** $|x^2-4x| = 3-2x$ **i** $|2x^2-2x+5| = 1-x$

4 Solve each of the following pairs of simultaneous equations.

a $y = x+4$
$y = |x^2-2|$

b $y = 1-x$
$y = |4x^2-4x|$

2.5 Graphs of $y = |f(x)|$ where $f(x)$ is linear

Exercise 2.5

1 Sketch the graphs of each of the following functions showing the coordinates of the points where the graph meets the axes.

a $y = |x-2|$ **b** $y = |3x-3|$ **c** $y = |3-x|$

d $y = \left|\frac{1}{3}x-3\right|$ **e** $y = |6-3x|$ **f** $y = \left|5-\frac{1}{2}x\right|$

2 **a** Complete the table of values for $y = 3-|x-1|$.

x	−2	−1	0	1	2	3	4
y		1		3			

b Draw the graph of $y = 3-|x-1|$ for $-2 \leqslant x \leqslant 4$.

3 Draw the graphs of each of the following functions.

a $y = |2x|+2$ **b** $y = |x|-2$ **c** $y = 4-|3x|$

d $y = |x-1|+3$ **e** $y = |3x-6|-2$ **f** $y = 4-\left|\frac{1}{2}x\right|$

4 Given that each of these functions is defined for the domain $-3 \leqslant x \leqslant 4$, find the range of

a $f : x \mapsto 6-3x$ **b** $g : x \mapsto |6-3x|$ **c** $h : x \mapsto 6-|3x|$.

5 **a** $f : x \mapsto 2 - 2x$ for $-1 \leqslant x \leqslant 5$

b $g : x \mapsto |2 - 2x|$ for $-1 \leqslant x \leqslant 5$

c $h : x \mapsto 2 - |2x|$ for $-1 \leqslant x \leqslant 5$

Find the range of each function for $-1 \leqslant x \leqslant 5$.

6 **a** Sketch the graph of $y = |3x - 2|$ for $-4 < x < 4$, showing the coordinates of the points where the graph meets the axes.

b On the same diagram, sketch the graph of $y = x + 3$.

c Solve the equation $|3x - 2| = x + 3$.

7 A function f is defined by $f(x) = 2 - |3x - 1|$, for $-1 \leqslant x \leqslant 3$.

a Sketch the graph of $y = f(x)$.

b State the range of f.

c Solve the equation $f(x) = -2$.

8 **a** Sketch on a single diagram, the graphs of $x + 3y = 6$ and $y = |x + 2|$.

b Solve the inequality $|x + 2| < \frac{1}{3}(6 - x)$.

2.6 Inverse functions

REMINDER

- **The inverse of the function** $f(x)$ is written as $f^{-1}(x)$.
- The domain of $f^{-1}(x)$ is the range of $f(x)$.
- The range of $f^{-1}(x)$ is the domain of $f(x)$.
- It is important to remember that not every function has an inverse.
- An inverse function $f^{-1}(x)$ can exist if, and only if, the function $f(x)$ is a one-one mapping.

WORKED EXAMPLE 5

$f(x) = (x + 3)^2 - 1$ for $x > -3$

a Find an expression for $f^{-1}(x)$.

b Solve the equation $f^{-1}(x) = 3$.

Answers

a $f(x) = (x + 3)^2 - 1$ for $x > -3$

Step 1: Write the function as $y =$ → $y = (x + 3)^2 - 1$

Step 2: Interchange the x and y variables. → $x = (y + 3)^2 - 1$

Step 3: Rearrange to make y the subject. → $x + 1 = (y + 3)^2$

$$\sqrt{x + 1} = y + 3$$

$$y = \sqrt{x + 1} - 3$$

$f^{-1}(x) = \sqrt{x + 1} - 3$

b $f^{-1}(x) = 3$.

$$\sqrt{x + 1} - 3 = 3$$

$$\sqrt{x + 1} = 6$$

$$x + 1 = 36$$

$$x = 35$$

Exercise 2.6

1 $f(x) = (x+2)^2 - 3$ for $x \geqslant -2$.
Find an expression for $f^{-1}(x)$.

2 $f(x) = \dfrac{5}{x-2}$ for $x \geqslant 0$.
Find an expression for $f^{-1}(x)$.

3 $f(x) = (3x-2)^2 + 3$ for $x \geqslant \dfrac{2}{3}$.
Find an expression for $f^{-1}(x)$.

4 $f(x) = 4 - \sqrt{x-2}$ for $x \geqslant 2$.
Find an expression for $f^{-1}(x)$.

5 $f : x \mapsto 3x - 4$ for $x > 0$ $\quad$ $g : x \mapsto \dfrac{4}{4-x}$ for $x \neq 4$.
Express $f^{-1}(x)$ and $g^{-1}(x)$ in terms of x.

6 $f(x) = (x-2)^2 + 3$ for $x > 2$
a Find an expression for $f^{-1}(x)$.
b Solve the equation $f^{-1}(x) = f(4)$.

7 $g(x) = \dfrac{3x+1}{x-3}$ for $x > 3$
a Find an expressions for $g^{-1}(x)$ and comment on your result.
b Solve the equation $g^{-1}(x) = 6$.

8 $f(x) = \dfrac{x}{2} - 2$ for $x \in \mathbb{R}$ $\quad$ $g(x) = x^2 - 4x$ for $x \in \mathbb{R}$
a Find $f^{-1}(x)$.
b Solve $fg(x) = f^{-1}(x)$ leaving answers as exact values.

9 $f : x \mapsto \dfrac{3x+1}{x-1}$ for $x \neq 1$ $\quad$ $g : x \mapsto \dfrac{x-2}{3}$ for $x > -2$
Solve the equation $f(x) = g^{-1}(x)$.

10 If $f(x) = \dfrac{x^2-9}{x^2+4}$ $x \in \mathbb{R}$ find an expression for $f^{-1}(x)$.

11 If $f(x) = 2\sqrt{x}$ and $g(x) = 5x$, solve the equation $f^{-1}g(x) = 0.01$.

12 Find the value of the constant k such that $f(x) = \dfrac{2x-4}{x+k}$ is a self-inverse function.

13 The function f is defined by $f(x) = x^3$. Find an expression for $g(x)$ in terms of x for each of the following:
a $fg(x) = 3x + 2$
b $gf(x) = 3x + 2$

TIP
A self-inverse function is one for which $f(x) = f^{-1}(x)$, for all values of x in the domain.

14 Given $f(x) = 2x + 1$ and $g(x) = \frac{x+1}{2}$ find the following:

a f^{-1} **b** g^{-1} **c** $(fg)^{-1}$ **d** $(gf)^{-1}$ **e** $f^{-1}g^{-1}$ **f** $g^{-1}f^{-1}$

Write down any observations from your results.

15 Given that $fg(x) = \frac{x+2}{3}$ and $g(x) = 2x + 5$ find $f(x)$.

16 Functions f and g are defined for all real numbers.
$g(x) = x^2 + 7$ and $gf(x) = 9x^2 + 6x + 8$. Find $f(x)$.

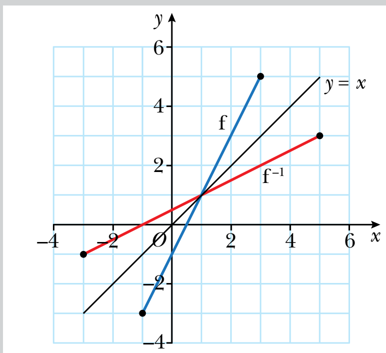
2.7 The graph of a function and its inverse

REMINDER

The graphs of f and f^{-1} are reflections of each other in the line $y = x$.

This is true for all one-one functions and their inverse functions.

This is because: $ff^{-1}(x) = x = f^{-1}f(x)$.



Some functions are called **self-inverse functions** because f and its inverse f^{-1} are the same.

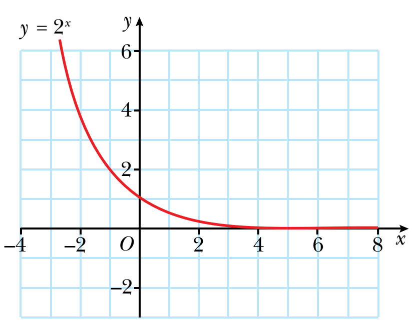
If $f(x) = \frac{1}{x}$ for $x \neq 0$, then $f^{-1}(x) = \frac{1}{x}$ for $x \neq 0$.

So $f(x) = \frac{1}{x}$ for $x \neq 0$ is an example of a self-inverse function.

When a function f is self-inverse, the graph of f will be symmetrical about the line $y = x$.

Exercise 2.7

1 On a copy of the grid, draw the graph of the inverse of the function $y = 2^{-x}$.



2 $f(x) = x^2 + 5,\ x \geqslant 0$.

On the same axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, showing the coordinates of any points where the curves meet the coordinate axes.

3 $g(x) = \frac{1}{2}x^2 - 4$ for $x \geqslant 0$.

Sketch, on a single diagram, the graphs of $y = g(x)$ and $y = g^{-1}(x)$, showing the coordinates of any points where the curves meet the coordinate axes.

4 The function f is defined by $f(x) = 3x - 6$ for all real values of x

a Find the inverse function $f^{-1}(x)$.

b Sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same axes.

c Write down the point of intersection of the graphs $f(x)$ and $f^{-1}(x)$.

5 Given the function $f(x) = x^2 - 2x$ for $x \geqslant 1$.

a Explain why $f^{-1}(x)$ exists and find $f^{-1}(x)$.

b State the range of the function $f^{-1}(x)$.

c Sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same axes.

d Write down where $f^{-1}(x)$ crosses the y axis.

6 **a** By finding $f^{-1}(x)$ show that $f(x) = \frac{3x-1}{2x-3}$ $x \in \mathbb{R},\ x \neq \frac{3}{2}$ is a self-inverse function.

b Sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same axes.

c Write down the coordinates of the intersection of the graphs with the coordinate axes.

Summary

Functions

A function is a rule that maps each x-value to just one y-value for a defined set of input values.

Mappings that are either $\left\{\begin{array}{l}\text{one-one}\\ \text{many-one}\end{array}\right.$ are called functions.

The set of input values for a function is called the **domain** of the function.

The set of output values for a function is called the **range** (or image set) of the function.

Modulus function

The modulus of x, written as $|x|$, is defined as

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

Composite functions

$fg(x)$ means the function g acts on x first, then f acts on the result.

$f^2(x)$ means $ff(x)$.

Inverse functions

The inverse of a function f(x) is the function that undoes what f(x) has done.

The inverse of the function f(x) is written as $f^{-1}(x)$.

The domain of $f^{-1}(x)$ is the range of f(x).

The range of $f^{-1}(x)$ is the domain of f(x).

An inverse function $f^{-1}(x)$ can exist if, and only if, the function f(x) is a one-one mapping.

The graphs of f and f^{-1} are reflections of each other in the line $y = x$.

Exercise 2.8

1 A one-one function f is defined by $f(x) = (x-2)^2 - 3$ for $x \geqslant k$.

- **a** State the least value that x can take.
- **b** For this least value of k, write down the range of f.

2 The function $f(x) = x^2 - 4ax$ (where a is a positive constant) is defined for all real values of x.

Given that the range is $\geqslant -8$, find the exact value of a.

3 $f(x) = (2x-1)^2 + 3$ for $x > 0$

$g(x) = \dfrac{5}{2x}$ for $x > 0$

Solve the equation $fg(x) = 7$.

4 The function f is defined by $f(x) = 1 - x^2$ for $x \in \mathbb{R}$.

The function g is defined by $g(x) = 2x - 1$ for $x \in \mathbb{R}$.

Find the values of x (in exact form) which solve the equation $fg(x) = gf(x)$.

5 Solve these simultaneous equations.

$y = 2x + 5$ $\qquad\qquad$ $y = \left|3 - x^2\right|$

6 **a** Sketch the graph of $y = |2x+1|$ for $-3 < x < 3$, showing the coordinates of the points where the graph meets the axes.

- **b** On the same diagram, sketch the graph of $y = 3x$.
- **c** Solve the equation $3x = |2x+1|$.

7 **a** Sketch the graph of $y = |x+3|$.

- **b** Solve the inequality $|x+3| > 2x + 1$.

8 $f(x) = x^2 - 3$ for $x \in \mathbb{R}$ $g(x) = 3x + 2$ for $x \in \mathbb{R}$

Solve the equation $gf(x) = g^{-1}(8)$.

9 Given the functions $f(x) = 2x + 3$ and $g(x) = \dfrac{1}{x+1}$ $x \in \mathbb{R}$ $x \neq 1$.

- **a** Find an expression for the inverse function $f^{-1}(x)$.
- **b** Find an expression for the composite function $gf(x)$.
- **c** Solve the equation $f^{-1}(x) = gf(x) - 1$.

10 Given the function $f(x) = \dfrac{2x+1}{x+2}$ $x \neq -2$.

- **a** Find $f^{-1}(x)$.
- **b** Find the points of intersection of the graphs of $f(x)$ and $f^{-1}(x)$.

Chapter 3: Simultaneous equations and quadratics

This section will show you how to:

- solve two equations where one is linear and one quadratic and interpret the results geometrically
- find the range and domain, maximum and minimum of quadratic functions and sketch their graphs
- solve quadratic inequalities
- understand the meaning of 'discriminant' and its relationship to the nature of the roots of a quadratic.

3.1 Simultaneous equations (one linear and one non-linear)

WORKED EXAMPLE 1

Solve the simultaneous equations.

$x + 2y = 4$

$x^2 + 4y^2 = 10$

Answers

$x + 2y = 4$ --------------(1)

$x^2 + 4y^2 = 10$ ---------(2)

From (1), $x = 4 - 2y$.

Substitute for x in (2):

$(4 - 2y)^2 + 4y^2 = 10$ expand brackets

$16 - 16y + 4y^2 + 4y^2 = 10$ rearrange

$8y^2 - 16y + 6 = 0$ simplify

$4y^2 - 8y + 3 = 0$ factorise

$(2y - 1)(2y - 3) = 0$

$y = \frac{1}{2}$ or $y = 1\frac{1}{2}$

Substituting $y = \frac{1}{2}$ into (1) gives $x = 3$.

Substituting $y = 1\frac{1}{2}$ into (1) gives $x = 1$.

The solutions are: $x = 1,\ y = 1\frac{1}{2}$ and $x = 3,\ y = \frac{1}{2}$.

Exercise 3.1

Solve the following simultaneous equations:

1 **a** $y = x^2 + 3x + 2$, $y = 2x + 8$

b $y = 3x^2 - 8x$, $y = 3x + 4$

c $x + y = 5$, $xy = 6$

d $y = x - 3$, $y^2 + xy + 4x = 7$

e $x^2 + y^2 = 5$, $y = \frac{2}{x}$

f $3y - 2x = 11$, $xy = 2$

g $y^2 + xy + 16x - 13 = 0$
$y = 3 - 2x$

h $2x^2 - xy + y^2 - 32 = 0$
$y = \frac{-5}{x}$

2 A rectangle is x cm long and y cm wide. Its area is 48 cm^2 and its perimeter is 32 cm.

a Use the above to form two equations.

b Find the dimensions of the rectangle.

3 A farmer uses 50 m of fencing to enclose a rectangular area against a long straight wall. What must be the dimensions of the enclosure if its area is to be 300 m^2.

4 I think of two positive numbers. If the difference between them is 12 and the sum of their squares is 314, find the numbers.

5 The straight line $y - 2x + 3 = 0$ intersects a curve with equation $y = x^2 - 2x$ at the points A and B. Find the coordinates of A and B.

6 The line $y = x + 5$ cuts the curve $y = x^2 - 3x$ at two points A and B. Find the coordinates of A and B and the exact value of the length of the straight line joining these points.

7 The line $y = ax - 1$ and the curve $y = x^2 + bx - 5$ intersect at the points $A(4, -5)$ and B. Find the values of a and b and the coordinates of B.

8 The straight line $2y - x = 5$ intersects with the circle $x^2 + y^2 = 25$ at A and B.

a Find the coordinates of A and B.

b Find the exact length of the line AB.

c Find the equation of the perpendicular line which passes through the midpoint of the line AB.

3.2 Maximum and minimum values of a quadratic function

REMINDER

The general rule is:

For a quadratic function $f(x) = ax^2 + bx + c$ that is written in the form:

$$f(x) = a(x - h)^2 + k$$

i if $a > 0$, the minimum point is (h, k)

ii if $a < 0$, the maximum point is (h, k).

Completing the square for a quadratic expression or function enables you to:

- write down the maximum or minimum value of the expression
- write down the coordinates of the maximum or minimum point of the function
- sketch the graph of the function
- write down the line of symmetry of the function
- state the range of the function.

WORKED EXAMPLE 2

$f(x) = 5 + 6x - 3x^2 \quad x \in \mathbb{R}$

a Find the value of a, the value of b and the value of c for which $f(x) = a - b(x + c)^2$.

b Write down the coordinates of the maximum point on the curve $y = f(x)$.

c Write down the equation of the axis of symmetry of the curve $y = f(x)$.

d State the range of the function $f(x)$.

Answers

a $5 + 6x - 3x^2 = a - b(x + c)^2$

$5 + 6x - 3x^2 = a - b\left(x^2 + 2cx + c^2\right)$

$5 + 6x - 3x^2 = a - bx^2 - 2bcx - bc^2$

Comparing coefficients of x^2, coefficients of x and the constant gives:

$-3 = -b$ --------(1) $\qquad 6 = -2bc$ -------(2) $\qquad 5 = a - bc^2$ -------(3)

Substituting $b = 3$ in equation (2) gives $c = -1$.

Substituting $b = 3$ and $c = -1$ in equation (3) gives $a = 8$.

So $a = 8$, $b = 3$ and $c = -1$.

b $y = 8 - 3(x - 1)^2$

This part of the expression is a square so it will always be $\geqslant 0$.
The smallest value it can be is 0.

The maximum value of the expression is $8 - 3 \times 0 = 8$ and this maximum occurs when $x = 1$.

So the function $y = 5 + 6x - 3x^2$ will have a maximum point at the point $(1, 8)$.

c The axis of symmetry is $x = 1$.

d The range is $f(x) \leqslant 8$.

Exercise 3.2

1 Express each of the following in the form: $(x - m)^2 + n$

a $x^2 + 6x - 1$ **b** $x^2 - 2x - 1$ **c** $x^2 - 3x + 1$ **d** $x^2 - x - 3$

2 Express each of the following in the form: $a(x - h)^2 + k$

a $2x^2 + 6x + 2$ **b** $2x^2 + 8x + 5$ **c** $3x^2 - 6x + 1$ **d** $2x^2 - x - 2$

3 Express each of the following in the form $m - (x - n)^2$.

a $10x - x^2$ **b** $12x - x^2$ **c** $5x - x^2$ **d** $7x - x^2$

4 Express each of the following in the form $a - (x + b)^2$.

a $5 - 4x - x^2$ **b** $8 - 6x - x^2$ **c** $12 - 5x - x^2$ **d** $9 - 3x - x^2$

5 Express each of the following in the form $a - p(x + q)^2$.

a $9 - 6x - 3x^2$ **b** $3 - 4x - 2x^2$ **c** $12 - 8x - 2x^2$ **d** $4 - 5x - 15x^2$

6 Given the function $f(x) = x^2 - 8x + 18, \ x \in \mathbb{R}$

a Write it in the form $f(x) = (x - p)^2 + q$ where p and q are integers to be found.

b Write down the equation of the axis of symmetry of the graph.

c Write down the coordinates of the minimum point of the graph of $f(x) = x^2 - 8x + 18$.

A quadratic function written in the form $f(x) = a(x - p)^2 + q$ with $a \neq 0$ has an axis of symmetry $x = p$ and a vertex (p, q).

7 Given the function $f(x) = -2x^2 - 12x + 7, \ x \in \mathbb{R}$.

a Express the function in the form $f(x) = a(x + p)^2 + q$ where p and q are integers to be found.

b Sketch the graph of the function.

c Write down the equation of its line of symmetry.

d Write down the maximum value of the function $f(x) = -2x^2 - 12x + 7$.

TIP: BE CAREFUL!
You need to distinguish between the maximum **value** of the function and the **coordinates** of the maximum value.

8 Given the function $f(x) = x^2 - 6x + 11, \ x \in \mathbb{R}$:

a Express the function in the form $f(x) = a(x - p)^2 + q$ where p and q are integers to be found.

b Write down the coordinates of the stationary point on the graph of $y = f(x)$.

c State whether this stationary point is a maximum or a minimum.

d Does the function meet the x axis?

e Sketch the function.

9 If $f(x) = 2x^2 - 12x + 23, \ x \in \mathbb{R}$

a Express the function in the form $f(x) = a(x - p)^2 + q$.

b Find the least value of $f(x)$ and the corresponding value of x.

c Write down the range of $f(x)$.

d Write down a suitable domain for $f(x)$ in order that $f^{-1}(x)$ exists.

10 Given that $5x^2 - ax + 14 \equiv b(x + 2)^2 + c$, find the values of a, b and c.

11 Express $f(x) = 2x^2 + 8x + 5$ in the form $f(x) = a(x + p)^2 + q$ stating the values of a, p and q.

12 a Express $f(x) = x^2 + 6x - 1$ in the form $f(x) = (x + p)^2 + q$.

b Hence solve the equation $x^2 + 6x - 1 = 0$ leaving your answers in exact form.

13 If $f(x) = 4x^2 + 6x - 12$ where $x \geqslant m$, find the smallest value of m for which f has an inverse.

14 $f(x) = 5x^2 - 3x + 7, \ 0 \leqslant x \leqslant 5$

a Express $5x^2 - 3x + 7$ in the form $a(x - b)^2 + c$ where a, b and c are constants.

b Find the coordinates of the turning point of the function $f(x)$, stating whether it is a maximum or minimum point.

c Find the range of f.

d State, giving a reason, whether or not f has an inverse.

3.3 Graphs of $y = |f(x)|$ where $f(x)$ is quadratic

REMINDER

To sketch the graph of the modulus function $y = |ax^2 + bx + c|$, you must:

- first sketch the graph of $y = ax^2 + bx + c$
- reflect in the x-axis the part of the curve $y = ax^2 + bx + c$ that is below the x-axis

WORKED EXAMPLE 3

a Find the coordinates of the stationary point on the curve $y = |(x+2)(x-2)|$.

b Sketch the graph of $y = |(x+2)(x-2)|$.

c Find the set of values of k for which $|(x+2)(x-2)| = k$ has four solutions.

Answers

a The x-coordinate of the stationary point is equidistant from the x-intercepts.

$|(x+2)(x-2)| = 0$

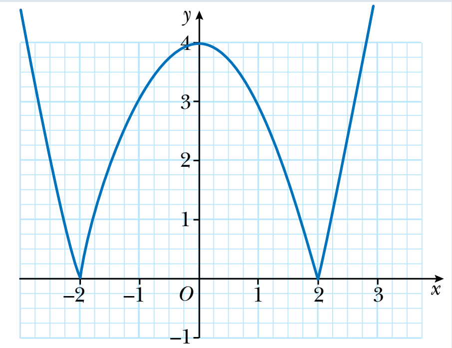
$x = -2$ or $x = 2$

The x-coordinate of the stationary point $= \dfrac{-2+2}{2} = 0$.

The y-coordinate of the stationary point $= |(0+2)(0-2)| = |-4| = 4$.

The minimum point is $(0, 4)$.

b



c $0 < k < 4$

Exercise 3.3

1 Sketch the graphs of each of the following functions.

a $y = |x^2 - x - 12|$ **b** $y = |x^2 + 3x - 4|$

c $y = |2x^2 - 5x - 3|$ **d** $y = |x^2 - 2x|$

2 $f(x) = 4 - 4x - x^2$

a Write $f(x)$ in the form $a - (x+b)^2$, where a, b and c are constants.

b Sketch the graph of $y = f(x)$.

c Sketch the graph of $y = |f(x)|$.